

New results for the dynamical critical behaviour of the two-dimensional Ising model

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Abstract

Using the new supercomputer JUMP at the Research Center Jülich, we were able to simulate large lattices (up to $L = 2 \cdot 10^6$, meaning a new world record) for long times (up to $T = 6000$ for $L = 1.5 \cdot 10^5$). Using this data, we examined the dynamical critical exponent z . The old assumption of $z = 2$ with logarithmic corrections seems very unlikely according to our data, leaving the asymptotic value of $z \simeq 2.167$.

Keywords: Ising model, Critical Point, Monte Carlo, Relaxation.

Introduction

Although the two-dimensional Ising model was solved exactly by Onsager in 1944 and much work has been done in this field over the last decades, some questions still remain open. One is the dynamical critical behaviour.

When we take a lattice with initially all spins up, right at the Curie point the magnetization M decays with time as $M \propto t^{-\beta/\nu z}$, where $\beta = 1/8$ is the exponent for the spontaneous magnetization and $\nu = 1$ is the exponent for the correlation length; both are known exactly.

Two main suggestions have been made for the value of z in Glauber kinetics: one is $z \simeq 2.167$ asymptotically, with simple power law behaviour ([1], [2], [3]), the other is $z = 2$ with logarithmic corrections to the power law behaviour. The latter was suggested by Domany [4] and later by Swendsen [5]. Lots of work has been done in order to rule out one of these assumptions (e. g. [7], [8], [6]), but with no final result.

We want to test these suggestions using new numerical data obtained by using the new supercomputer JUMP at the Research Center Jülich. We compare these with older data.

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Simulations

Data was generated by initializing a lattice with all spins up and then doing a Monte Carlo simulation of the Ising model with Glauber kinetics. To obtain higher speed, multi spin coding and parallelization (via MPI) were used. Speed on one 1.7 GHz Power4+ processor was roughly 160 million sites per second. Up to 512 processors were used in parallel for the largest simulations.

Although the new supercomputer JUMP is rather large, there are some restrictions on the size of lattices that can be simulated. We have chosen $L = 1.5 \cdot 10^5$ and $L = 2 \cdot 10^6$ (the old world record for the two-dimensional Ising model was $L = 10^6$ [10], [3]), with periodic boundary conditions. Thus finite size effects should be negligibly small. Several independent runs were done for $L = 1.5 \cdot 10^5$ for averaging, 50 runs each for the random number generators $x_{n+1} = 13^{13} \cdot x_n \bmod 2^{63}$ (called LCG(13¹³)) and the 64-bit implementation of Ziff's four-tap generator R(471,1586,6988,9689) [9]. For $L = 1.5 \cdot 10^5$ the simulations were done up to 6000 timesteps (full sweeps through the lattice). For the larger lattices, only considerably smaller times were possible, due to restrictions in computing time. For investigating finite-size effects, lattices with $L = 5 \cdot 10^4$ were simulated with LCG(16807), again averaging over 50 runs.

The effective exponent z can be determined from the $M(t)$ data by numerical differentiation: $-1/8z = d(\log M)/d(\log t)$.

Results

Even when averaging $M(t)$ over several independent runs, fluctuations are visible when calculating the effective $z(t)$. Thus each point in Figs. 1–4 represents many $z(t)$; these points were generated by dividing the data for $z(1 \dots 6000)$ into several intervals and then doing a least squares fit in each. Each point is the central point of the fit in the interval. The errorbars for z (not shown in the plots for better legibility) are of the order of the symbol size for short times and grow to up to ± 0.03 for long times. The new data, especially for the large systems, allows for a better fit of the Swendsen suggestion. The new fitted curve has a maximum at about $t \simeq 1700$ ($1/t \simeq 6 \cdot 10^{-4}$).

The last point for $L = 1.5 \cdot 10^5$, corresponding to the interval $t = 3000 \dots 6000$, is subject to strong fluctuations and thus doubtful. Unfortunately, this is the most interesting data point. Nevertheless, a trend is visible: for larger times, the critical exponent seems to go up, not down, thus being in contradiction to the Domany-Swendsen suggestion.

This could also be due to finite-size effects: for $L = 5 \cdot 10^4$, the effect of increasing z seems to be stronger (cf. Fig. 4), but more simulations would be needed for confirmation.

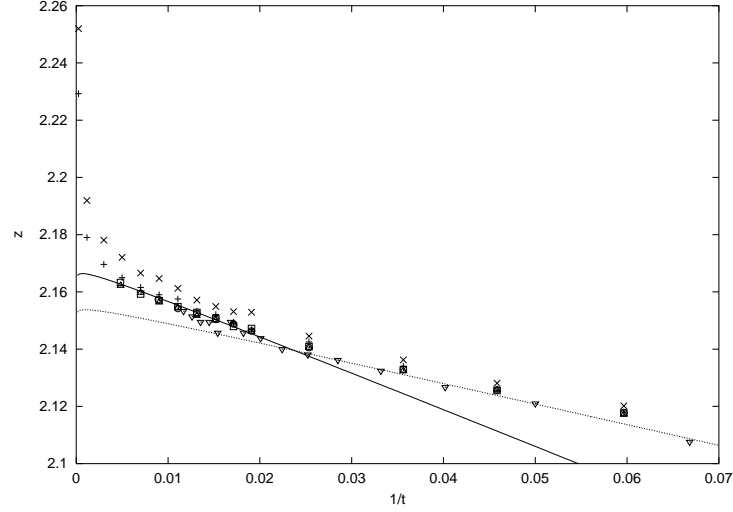


Figure 1: Monte Carlo data for the two-dimensional Ising model at the Curie point. $+$ are for $L = 1.5 \cdot 10^5$ with LCG(13¹³) generator, averaged over 50 runs, \times for $L = 1.5 \cdot 10^5$ with R(471,1586,6988,9689) generator, averaged over 50 runs, \square for $L = 2 \cdot 10^6$ with LCG(13¹³), \circ for $L = 2 \cdot 10^6$ with R(471,1586,6988,9689), \triangle for $L = 2 \cdot 10^6$ with LCG(16807), ∇ for $L = 10^6$ with LCG(16807), data by Stauffer [3] (large systems one run each). The lines represent the Swendsen suggestion for a fit, $\beta / \left(\frac{t}{t-t_0} \cdot \left(\frac{1}{2\beta} - \frac{c}{1+c \cdot \log(t-t_0)} \right) \right)$, with $c = 0.005$ and $t_0 = 0.6$ for the solid line (new fit parameters) and $c = 0.004625$ and $t_0 = 0.34$ for the dotted line (Swendsen's original parameters [5]).

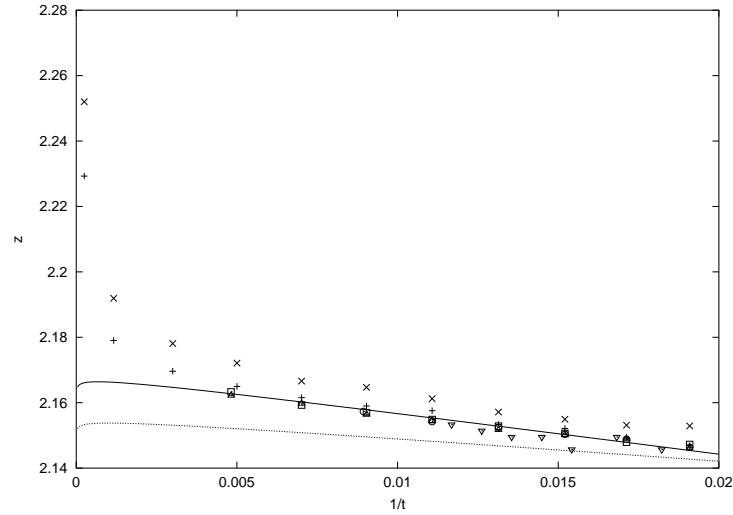


Figure 2: Same plot as in fig. 1, but with expanded $1/t$ -axis.

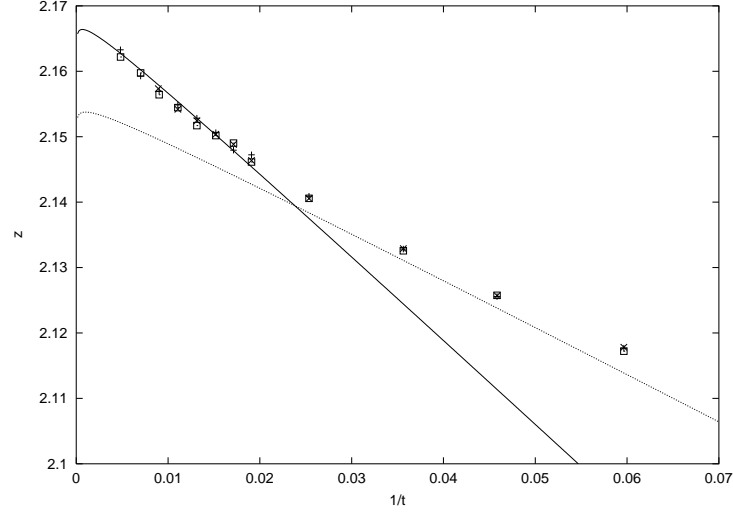


Figure 3: Monte Carlo data for three runs with $L = 2 \cdot 10^6$. + are for LCG(13^{13}), \times for R(471,1586,6988,9689), \square for LCG(16807); data is the same and the lines represent the same fits as in figs. 1 and 2.

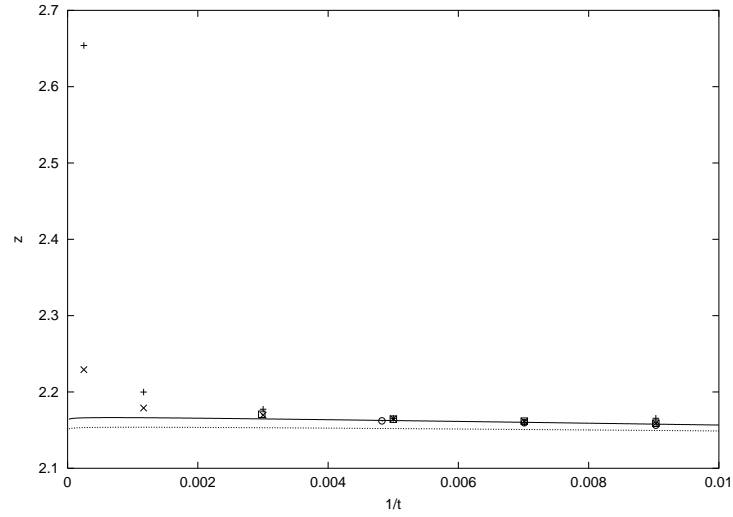


Figure 4: Monte Carlo data for $L = 5 \cdot 10^4$ with LCG(16807), averaged over 50 runs (+), $L = 1.5 \cdot 10^5$ with LCG(13^{13}), averaged over 50 runs (\times), $L = 10^6$ with R(471,1586,6988,9689), averaged over four runs (\square), and $L = 2 \cdot 10^6$ with LCG(16807), one run (\circ).

Conclusion

Although it is possible to argue that the numerical data for very long times is doubtful, as the influence of fluctuations on the value of z increases, the current precision data seems bad for the Domany-Swendsen assumption. It would be possible to modify the fit to the data, but the trend for long times rather contradicts the value of $z = 2$.

Nevertheless, there is still work to do: the influence of various random number generators is important, in order to find one which allows precise data, but is still fast enough for large-scale simulations: for $L = 1.5 \cdot 10^5$, averaged over 50 runs, Ziff's four-tap generator produces results which differ systematically from other generators and system sizes. This is not the case for a single run with $L = 2 \cdot 10^6$ and four-tap generator. Data for $L = 5 \cdot 10^4$, averaged over 25 runs, showed the same behavior as for $L = 1.5 \cdot 10^5$. For these lattice sizes, R(471,1586,6988,9689) seems not to be suited well.

Furthermore, the influence of finite-size effects for long times should be investigated in more detail, and the effects of fluctuations in the magnetization in general.

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